New types of homotopy bialgebras in Geometry and Topology

→ Joint work with Johan Leray: Arxiv: 2203.05062
see also Kontsevich – Takeda – Vlassopoulos: Arxiv: 2112.14667
and Alexandre Quesney: Arxiv: 2312.14893

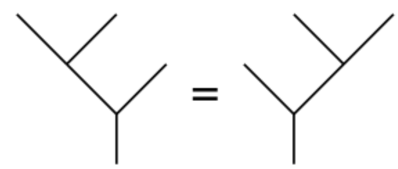
(1) State of the art: A_∞- algebras

. Algebraic structures to detect Topology: Homotopy invariants.

X: topological space k: ring (field of characteristic 0).

→ cochain algebra of X: $(C^\bullet_{sing}(X,k), d, \cup)$ cup product: associative

$\Downarrow$



cohomology algebra $(H^\bullet_{sing}(X,k), \bar{\cup})$ again associative

2 but for a "bad" reason.

. How invariants are these constructions?

$X \xrightarrow{\sim} Y$ in Top
homotopy equivalence

$$\Longrightarrow \begin{cases} C^\bullet(Y,k) \xrightarrow[\text{quasi-iso morphism}]{\sim} C^\bullet(X,k) \\ H^\bullet(Y,k) \xrightarrow[\text{iso}]{\cong} H^\bullet(X,k) \end{cases}$$

in dg associative algebras

↳ Understand the homotopy theory of dg associative algebras up to quasi-iso morphisms.

→ dga alg $[q.i.^{-1}]$ localisation

$A \to \cdot \overset{\sim}{\leftarrow} \cdot \to \cdot \overset{\sim}{\leftarrow} \cdot \quad \cdots\cdots \quad \cdot \to B$

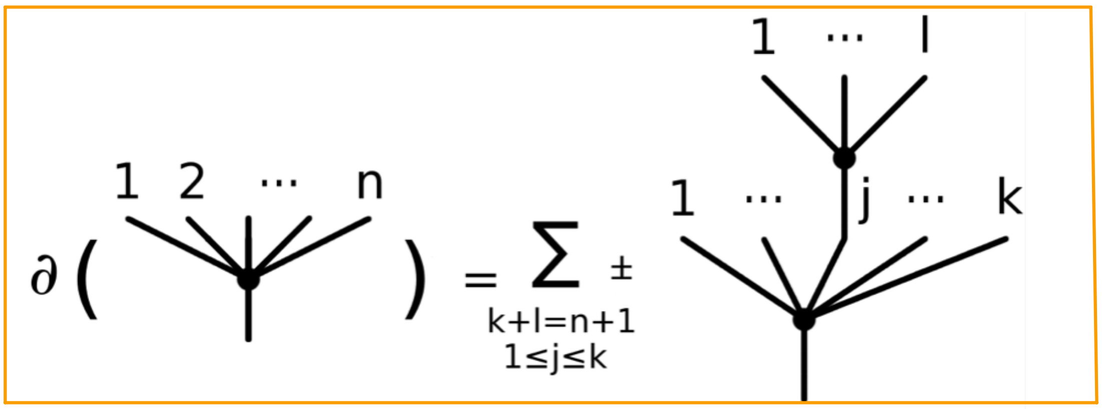
$\not\exists$ algebro-homotopical data lost with $(H^\bullet(X, k), \bar{\cup})$.

The solution [Stasheff, Kadeishvili] Associative algebras up to homotopy aka A_∞-algebras

Def: [A_∞-algebra] $(\underbrace{A, d}_{\text{chain complex}}, \mu_2, \mu_3, \mu_4, \ldots, \mu_n, \ldots)$

$\mu_n : A^{\otimes n} \longrightarrow A \qquad |\mu_n| = n-2$

st

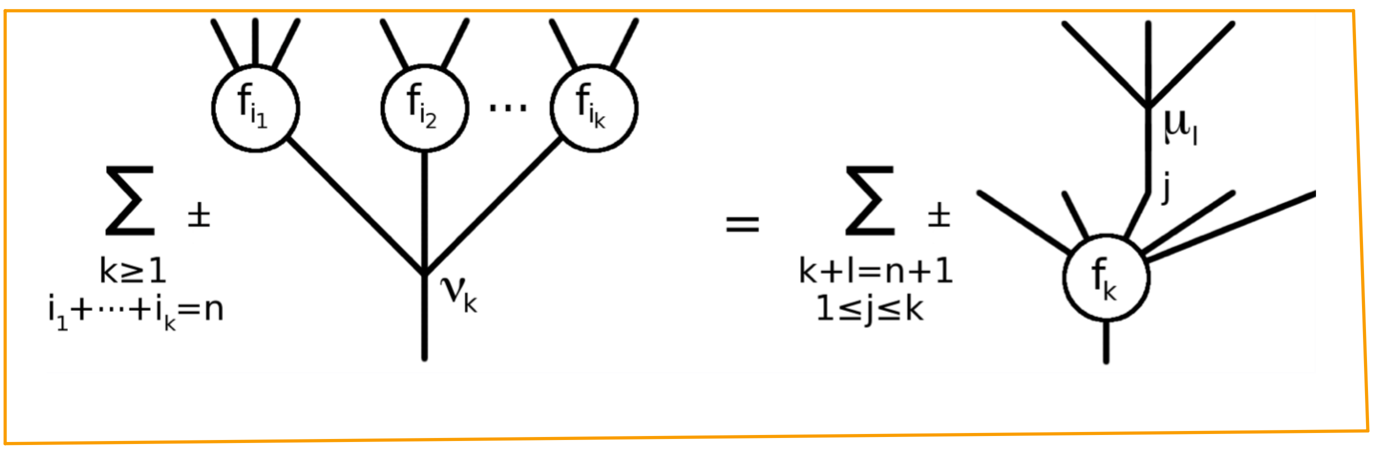


$\longleftrightarrow$ Maurer-Cartan element in the "Hochschild" cochain complex $\prod_n \mathrm{Hom}(A^{\otimes n}; A)$.

$\longrightarrow$ Higher notion of morphisms

Def: [A_∞-morphisms] $(A, \mu) \rightsquigarrow (B, \nu)$: collection of maps: $\{ f_n : A^{\otimes n} \longrightarrow B,\ |f_n| = n-1 \}$

st



• A_∞-quasi isomorphism: when f_1 is a quasi-isomorphism

$\boxed{\text{Thm}}$ [Homotopy transfer] Given a deformation retract

$$h \circlearrowright (A, d_A) \underset{i}{\overset{p}{\rightleftarrows}} (H, d_H)$$

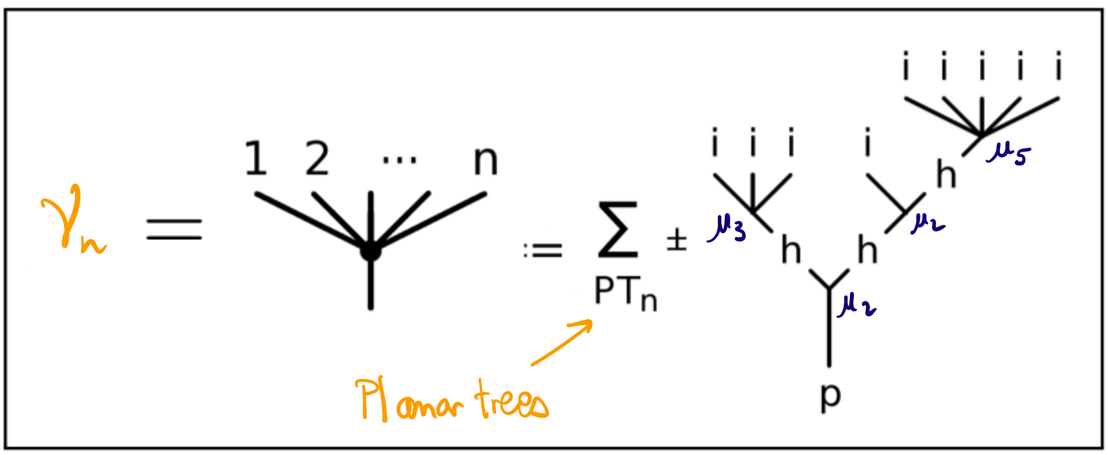
and an A_∞-algebra structure μ on A

$\hookrightarrow$ there exists

$$h_\infty \circlearrowright \underset{\mu}{(A, d_A)} \underset{i_\infty}{\overset{p_\infty}{\rightleftarrows}} \underset{\nu}{(H, d_H)}$$

- an A_∞-algebra structure ν on H,
- an extension of i into an A_∞-quasi-isomorphism,
- an extension of p into an A_∞-quasi-isomorphism,
- an extension of h into an A_∞-homotopy.

$\hookrightarrow$ Comes with explicit formulas:



Example: $H^\bullet(X, k)$: $\nu_2 = \smile$

but ν_n "=" Massey products: no loss of data

Thm Any A_∞- quasi-isomorphism $A \xrightarrow{\sim} B$ admits a "homotopy" inverse $B \xrightarrow{\sim} A$.

$\hookrightarrow$ $\exists$ zig-zag of quasi-isomorphisms $\iff$ $\exists$ an A_∞-quasi-isomorphism

$A \xleftarrow{\sim} \cdot \xrightarrow{\sim} \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xrightarrow{\sim} B$ $\qquad$ $A \xrightarrow{\sim} B$

Application: formality theorems

Thm [C-RN-P-W] The dg associative algebra structure $(C^\bullet_{sing}(X, \mathbb{Q}), d, \cup)$ detects faithfully the rational homotopy type of X (connected, nilpotent, based, finite type)

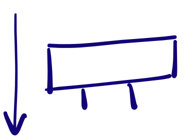
② Beyond Associative / A_∞ - algebras

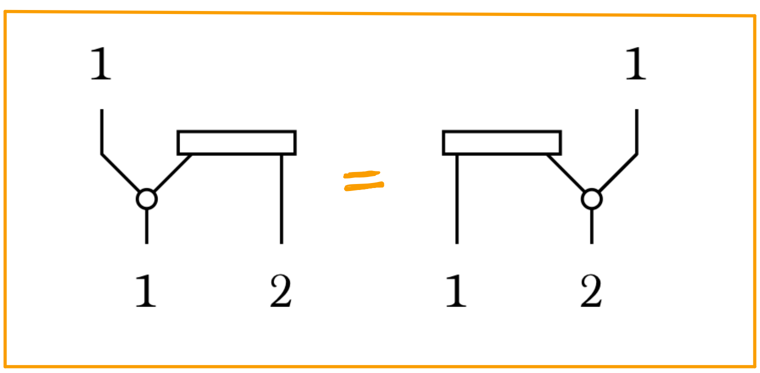
$\rightarrow$ Encode Poincaré duality : X oriented manifold

Thom class $\Rightarrow$ $\exists\, c \in \left(H^\bullet(X)^{\otimes 2}\right)^{S_2}$

Def [V_algebra] (A, d, μ) associative algebra equipped with

$c \in (A^{\otimes 2})^{S_2}$


$: A^{\otimes 0} \to A^{\otimes 2}$ s.t.



Ex: $(H^\bullet(X, k), \bar{\cup}, c)$: V_algebra $\qquad$ X: oriented manifold.

→ Non-commutative Poisson geometry

• classical geometry

$C^\infty(M)$: commutative algebra

• non-commutative geometry

A : associative algebra

Kontsevich - Rosenberg ← principle (affine scheme of representations)

• Poisson geometry

$\pi \in \Gamma(\Lambda^2 TM)$, $[\pi,\pi]=0$

$\Updownarrow$

$\{;\} : C^\infty(M) \wedge C^\infty(M) \to C^\infty(M)$

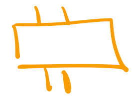
Lie bracket

$(C^\infty(M), \cdot, \{,\})$ Poisson algebra

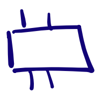
• NC Poisson geometry

NC bivector field s.t. Maurer - Cartan equation

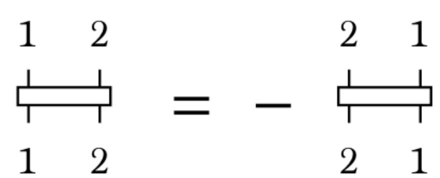
$\Updownarrow$

double bracket :  : $A^{\otimes 2} \to A^{\otimes 2}$

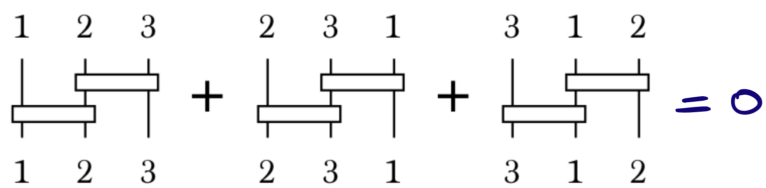
[Van den Bergh]

Def [Double Poisson gebra] $(\underbrace{A, \mu}_{\text{associative algebra}},$  $)$ s.t.

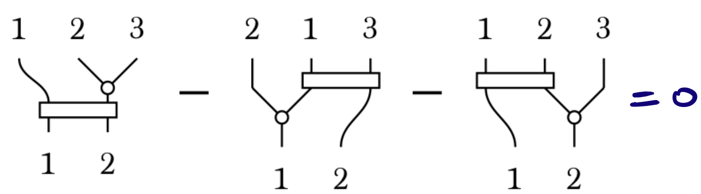
• cyclic symmetry :



• double Jacobi relation :



• "Leibniz" relation :



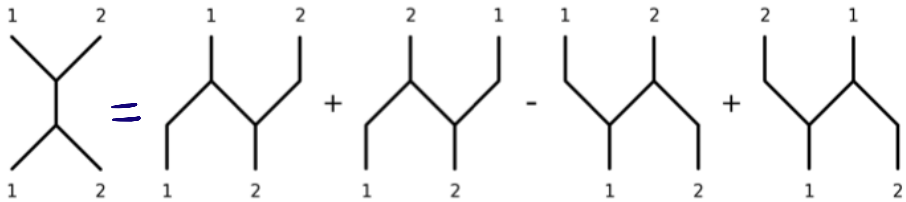
→ Quantum algebra, discrete geometry

$C^\infty(M)$: functions on Poisson-Lie group: Lie bialgebra [Drinfeld]

Y : $[\,;\,]$ Lie bracket

λ : δ : Lie cobracket

satisfying



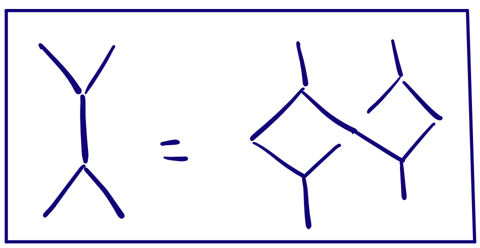
Classical

↓ quantisation of Lie bialgebras [Etingof-Kazhdan]

quantum : associative bialgebras Y associative

λ coassociative

s.t. λ is a morphism :



↓ infinitesimal version

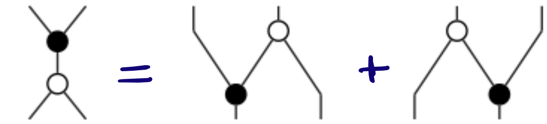
derivation

Def [Joni-Rota, Aguiar]

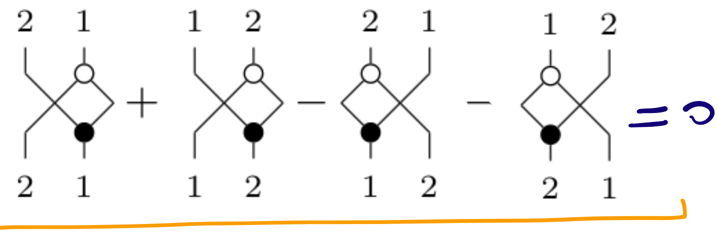
- Infinitesimal bialgebra : (A, Y, λ)

associative coassociative

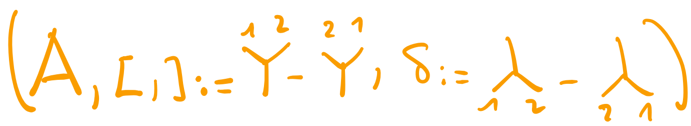
s.t. λ derivation :



• <u>balanced</u> when



$\Downarrow$

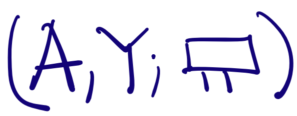


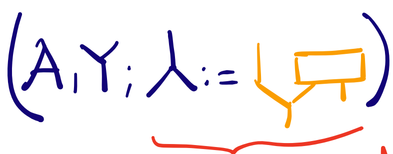
: Lie bialgebra

Application: [Aguiar] cd-index of polytopes.

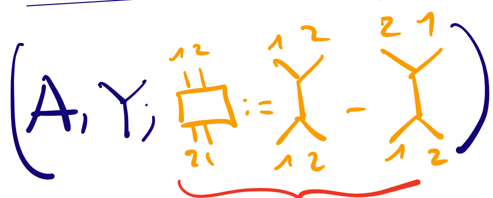
③ Comparison

V_algebra $\not\to$ Inf. Balanced bialgebra $\not\to$ double Poisson gebra





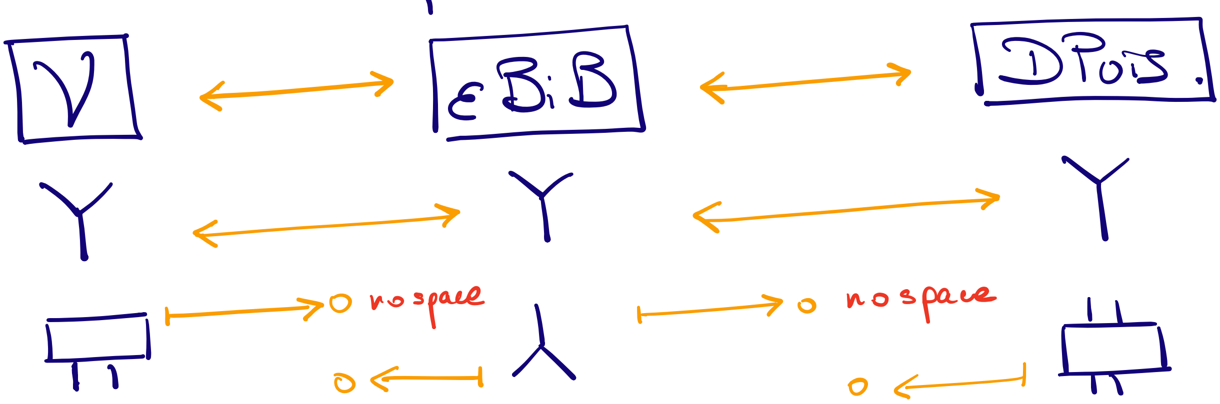
not a derivation



does not satisfy the Leibniz relation

all three notions encoded by properads (or dioperads)

= "operads with multiple inputs & outputs"



→ Only trivial statements like
"any double Poisson gebra $(A, Y; \boxed{\text{⊓⊔}})$ carries a canonical infinitesimal balanced bialgebra: $(A, Y, \circ)$", i.e their sole common point is Y.

$\boxed{\text{Thm}}$ [Leray-V., Quesney]

- $\exists$ canonical epimorphisms of dg properads

"genuso"

$$\boxed{\mathcal{V}_\infty = pC\mathcal{Y} \twoheadrightarrow \varepsilon BiB_\infty \twoheadrightarrow DPois_\infty}$$

↳ i.e their homotopy bialgebra versions carry more and more structural operations!

Proof: The three properads $\mathcal{V}$, εBiB, and $DPois$ are quadratic

$\Downarrow$ Definition

$$\mathcal{V}_\infty := \Omega\, \mathcal{V}^{¡} \qquad \varepsilon BiB_\infty := \Omega\, \varepsilon BiB^{¡} \qquad DPois_\infty := \Omega\, DPois^{¡}$$

Koszul dual coproperad

cobar construction (free properad)

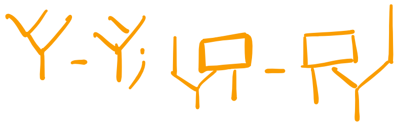
coproperads → properads

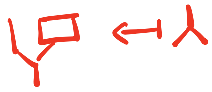
connected graphs

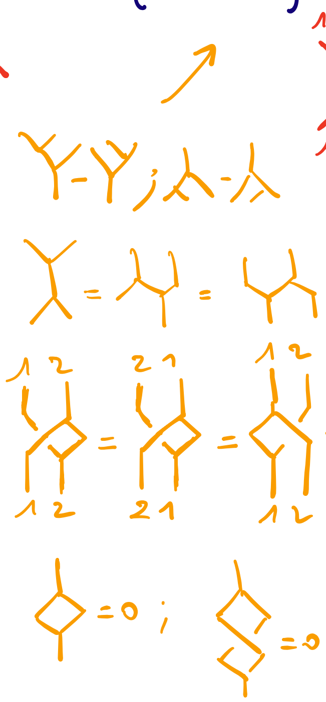
$$\mathcal{P} = \mathcal{G}(V)/(R) \qquad \underbrace{\mathcal{P}^! := \mathcal{G}(V^*)/(R^\perp)}_{\text{Koszul dual properad}} \qquad \mathcal{P}^{¡} := (\mathcal{P}^!)^*$$

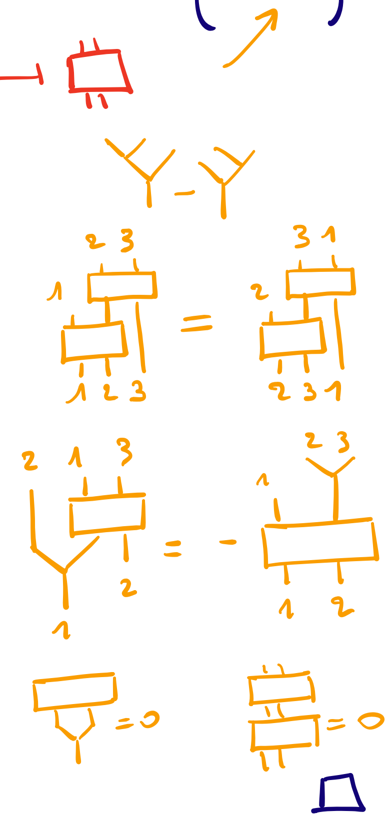
$$R \subset \mathcal{G}(V)^{(2)}$$

$$\mathcal{V}^! = \frac{\mathcal{G}(Y, \square)}{(\quad)} \cong \mathcal{V} \longleftarrow \varepsilon BiB^! = \frac{\mathcal{G}(Y;\lambda)}{(\quad)} \longleftarrow DPois^! = \frac{\mathcal{G}(Y; \square)}{(\quad)}$$









→ "More degrees of freedom on the Koszul dual level"

Rk:

- [Poirier–Tradler] $\mathcal{V}_\infty$-gebras = pre-Calabi-Yau algebras [K-T-V]
  The dioperad $\mathcal{V}_{g=0}$ is Koszul: is the properad $\mathcal{V}$ Koszul?
- [Leray] The properad DPois is Koszul.
- Is the dioperad / properad εBiB Koszul?

Thm [Leray-V., Quesney]

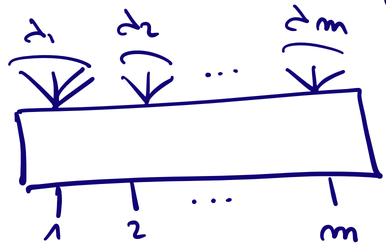
- $\exists$ canonical isomorphisms of properads

$$\boxed{c\mathcal{V}_\infty \xrightarrow{\cong} c\varepsilon BiB_\infty \xrightarrow{\cong} c\,DPois_\infty}$$

Versions with curvatures

↳ Interpretation: equivalence of the two definitions of derived NC Poisson geometry

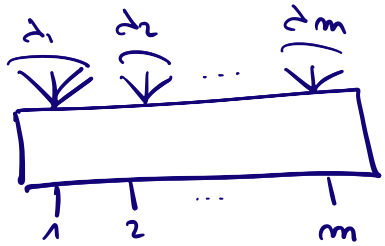
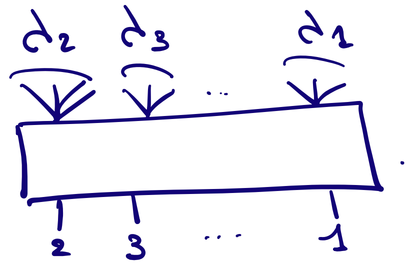
Definition [Homotopy curved $\mathcal{V}$-εBiB-DPois-gebra]

(A,d) chain complex equipped with  $: A^{\otimes n} \longrightarrow A^{\otimes m}$

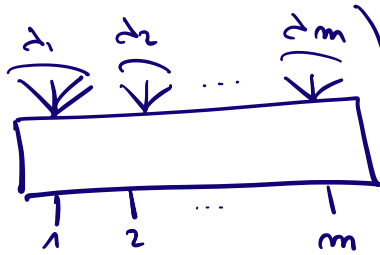
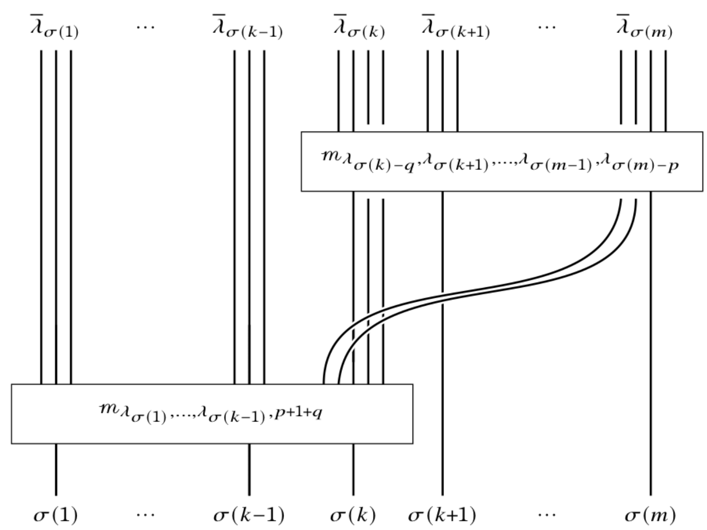
$n := \lambda_1 + \dots + \lambda_m$

degree $= n-2$

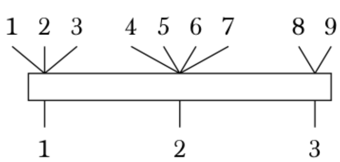
$m \geq 1$
$\lambda_1, \dots, \lambda_m \geq 0$

- cyclic symmetry:  $= \pm$ .

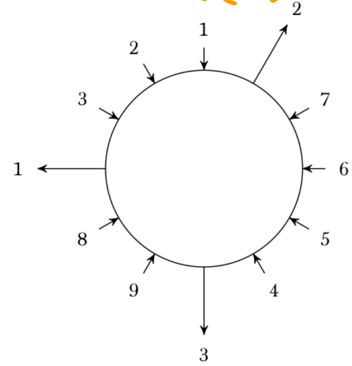
- quadratic relation:

$\partial\Big($  $\Big) = \sum_{k=1}^{m} \sum_{\sigma \in \mathbb{C}_m} \sum_{\substack{0 \leqslant p < \lambda_{\sigma(m)} \\ 0 \leqslant q < \lambda_{\sigma(k)}}} \pm$ 

Better: curved cyclic A_∞-algebra structure on $A \oplus A^*$ s.t. $\mu_n|_{(A^*)^{\otimes n}} \equiv 0$ [Seidel]

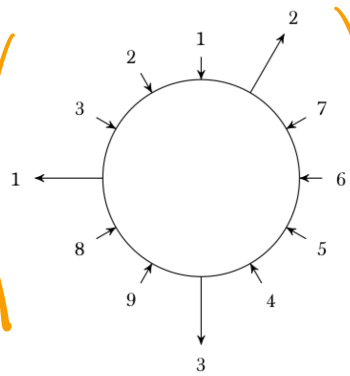
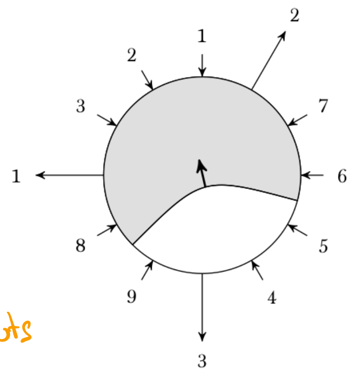
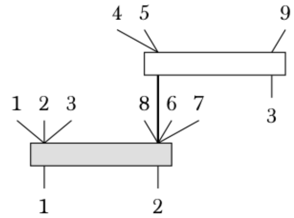


$\xleftrightarrow[\text{symmetry}]{\text{cyclic}}$



"roundabouts"

↳ Quadratic relation: [KTV]

$\partial\Bigg($  $\Bigg) = \sum_{\substack{\text{partitions} \\ \text{of roundabouts} \\ \text{into 2, ie} \\ \text{by an arc}}} \pm$  $=$ 

Definitions

- $pCY = V_\infty$-gebras: add the restriction $d_1 \geqslant 2$ for $m=1$ (genus 0)

$\cup$

- εBiB_∞-gebras: add the restriction $n>0$

$\cup$

- $DPois_\infty$-gebras: add the restriction $d_1, \dots, d_m \geqslant 1$

④ Homotopy theory of homotopy gebras

- All the homotopical properties of A_∞-algebras hold true for operads of the form $\mathcal{P}_\infty = \Omega\, \mathcal{C}$ ($\mathcal{C}$ cooperad):

∞-morphisms, homotopy transfer theorem, homotopy invertibility of ∞-quasi-isomorphisms [now well known → book Loday-V.]

↳ "NEW": also hold true for $\mathcal{P}_\infty$-gebras with $\mathcal{P}_\infty = \Omega\, \mathcal{C}$ (coproperad)

- with explicit formulas [Hoffbeck-Leray-V., '20]

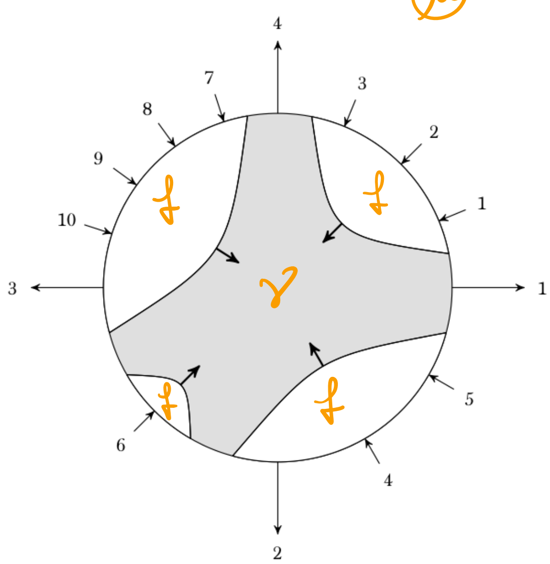
Examples: The 3 above properads: $\mathcal{V}$, εBiB, $DPois$.

- Definition $\Longleftrightarrow$ Maurer-Cartan element in a higher Hochschild complex [KTV]

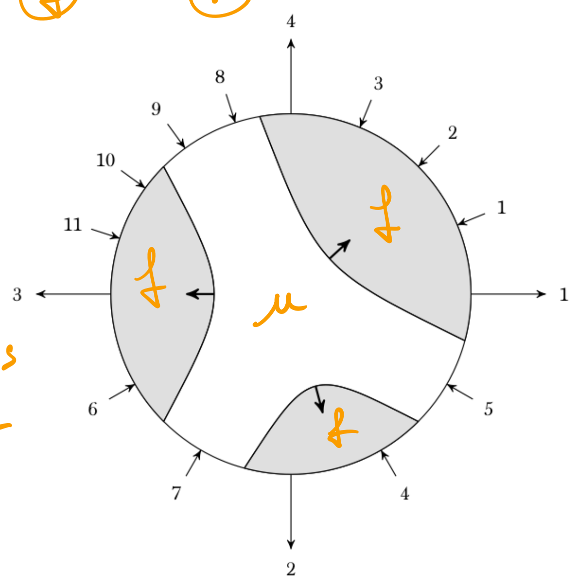
- **∞-morphisms**: encoded by roundabouts

$$\underset{\mu}{A^{\otimes d_1+\dots+d_m}} \xrightarrow{\quad f \quad} \underset{\gamma}{B^{\otimes m}} \quad \text{st.}$$

$$\sum_{\substack{\text{partitions}\\ \text{1-black}}} \pm$$

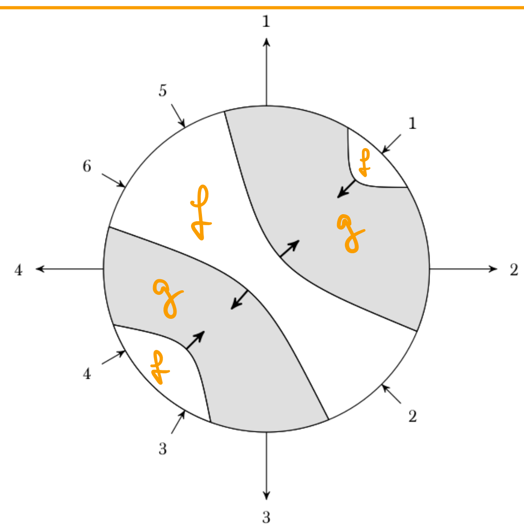


$$= \sum_{\substack{\text{partitions}\\ \text{1-white}}} \pm$$



- **Composite of ∞-morphisms**

$$g \circ f = \sum_{\substack{\text{bicolored}\\ \text{partitions.}}} \pm$$



Thm [Homotopy transfer, Hoffbeck-Leray-V.'20]

deformation retract $h \circlearrowright (A, d_A) \underset{i}{\overset{p}{\rightleftarrows}} (H, d_H)$ and

an $\mathcal{P}_\infty$-gebra structure: μ on A

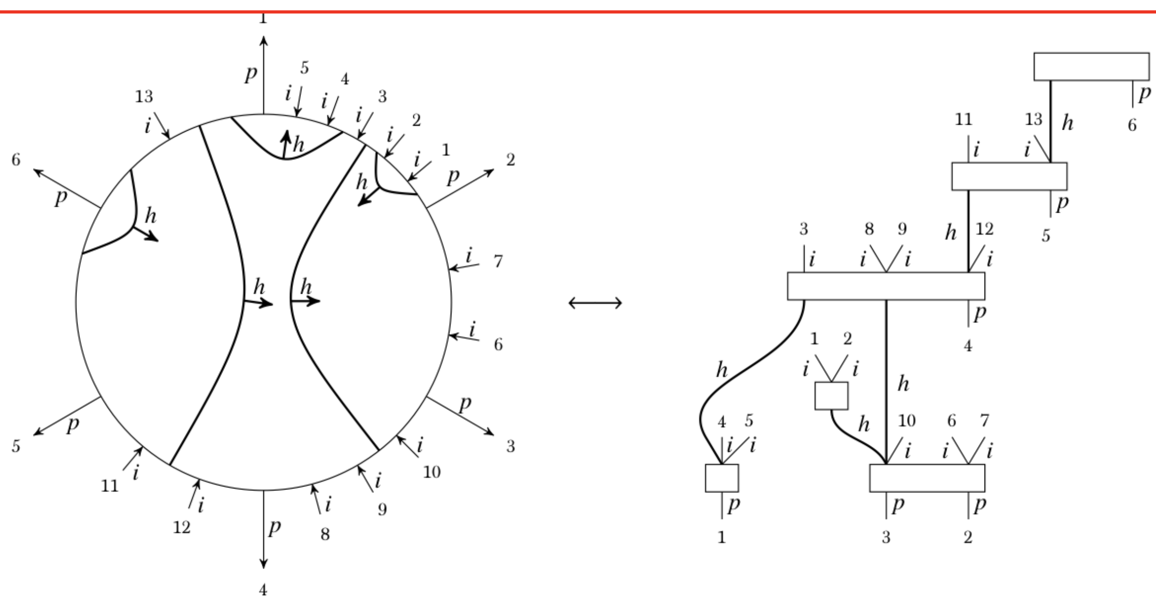
↳ there exists

$$h_\infty \circlearrowright \underset{\mu}{(A, d_A)} \underset{i_\infty}{\overset{p_\infty}{\rightleftarrows}} \underset{\nu}{(H, d_H)}$$

- a $\mathcal{P}_\infty$-gebra structure ν on H,
- an extension of i into an ∞-quasi-isomorphism,
- an extension of p into an ∞-quasi-isomorphism.

- <u>Explicit formulas</u>:

$$\nu = \sum_{\text{partitions}}$$



2 genus 0 here: more complicated in general (graphs with levels)

Thm [H-L-V.'24]

$\boxed{\text{Any } \infty\text{-quasi-isomorphism } A \rightsquigarrow B \text{ admits a "homotopy" inverse } B \overset{\sim}{\rightsquigarrow} A.}$

↳ $\exists$ zig-zag of quasi-isomorphisms $A \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdot \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xrightarrow{\sim} B$ $\Longleftrightarrow$ $\exists$ an ∞-quasi-isomorphism $A \overset{\sim}{\rightsquigarrow} B$

<u>Application:</u> $\underbrace{\text{new formality theorem}}_{\text{your turn to play!}}$

$\boxed{\text{Thank You!}}$